Appendix of Chapter 4
Additional Material about the Separation of Variable

4-5. Separation of Variables. Steady Two-Dimensional Cartesian Geometry

From Arpaci, "Conduction Heat Transfer"

When the boundary conditions of a problem are in terms of specified T, $\partial T/\partial n$, or $\partial T/\partial n + BT$, where n is the normal to the boundary and B a constant, the solution may be expressed as a product of functions of each coordinate separately. This allows the boundary conditions to be expressed in terms of a single variable, and reduces the partial differential equation to a set of ordinary differential equations.

The essential features of the method will now be illustrated by means of a steady two-dimensional example. Consider the second-order partial differential equation

$$a_1(x)\frac{\partial^2 T}{\partial x^2} + a_2(x)\frac{\partial T}{\partial x} + a_3(x)T + b_1(y)\frac{\partial^2 T}{\partial y^2} + b_2(y)\frac{\partial T}{\partial y} + b_3(y)T = 0. \quad (4\text{-}41)$$

A more generalized form of this equation which involves coefficients as functions of both independent variables is not suitable for the separation of variables.

Assume the existence of a product solution

$$\underline{T(x, y) = X(x)Y(y),} \quad (4\text{–}42)$$

where X is a function of x alone and Y is a function of y. This assumption becomes meaningful when the two functions X and Y actually satisfy separate differential equations.

Introducing Eq. (4–42) into Eq. (4–41) and dividing the result by XY yields

$$\left[a_1(x)\frac{d^2X}{dx^2} + a_2(x)\frac{dX}{dx} + a_3X\right]\frac{1}{X} = -\left[b_1(y)\frac{d^2Y}{dy^2} + b_2(y)\frac{dY}{dy} + b_3(y)Y\right]\frac{1}{Y}$$
$$= \pm\lambda^2 \quad (4\text{–}43)$$

Hence the partial differential equation of Eq. (4–41) is reduced to the following two ordinary differential equations:

$$a_1(x)\frac{d^2X}{dx^2} + a_2(x)\frac{dX}{dx} + [a_3(x) \pm \lambda^2]X = 0,$$
$$b_1(y)\frac{d^2Y}{dy^2} + b_2(y)\frac{dY}{dy} + [b_3(y) \mp \lambda^2]Y = 0. \quad (4\text{–}44)$$

★ The method of separation of variables is applicable to steady two-dimensional problems if and when (i) *one of the directions of the problem is expressed by a homogeneous differential equation subject to homogeneous boundary conditions (the homogeneous direction), while the other direction is expressed by a homogeneous differential equation subject to one homogeneous and one nonhomogeneous boundary condition (the nonhomogeneous direction)*, and (ii) *the sign of λ^2 is chosen such that the boundary-value problem of the homogeneous direction leads to a characteristic-value problem.*

The solutions obtained by the separation of variables are in the form of a sum or integral, depending on whether the homogeneous direction is finite or extends to infinity, respectively.

4–7. Nonhomogeneity From Arpaci, "Conduction Heat Transfer."

So far, the steady two-dimensional cartesian problems that we have solved by the method of separation of variables were those involving a homogeneous differential equation subject to two homogeneous boundary conditions in the finite direction, and one homogeneous plus one nonhomogeneous boundary condition in the remaining (finite or infinite) direction. Most two-dimensional problems,

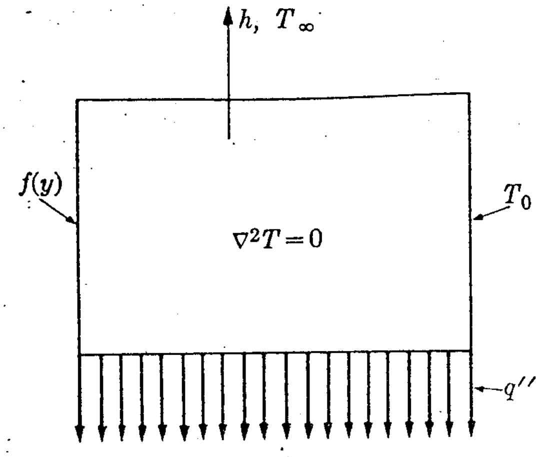

FIG. 4–25

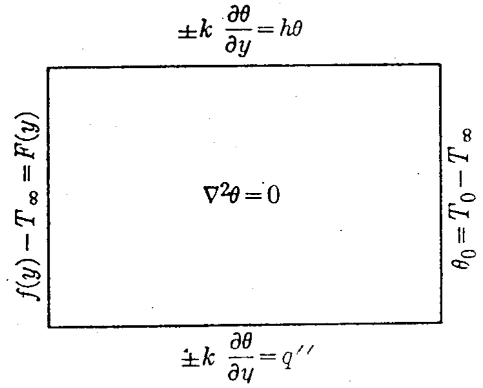

FIG. 4–26

however, do not *a priori* satisfy these conditions. In such cases the problem must be transformed, if possible, to one that does satisfy them. One way of doing this is to shift the temperature level, a procedure which has been employed with previous examples in this chapter. If this is impossible, then using the principle of superposition, we may divide the problem into a number of simpler problems each satisfying the required homogeneity conditions and all adding up to the posed problem.

Nonhomogeneities may result from nonhomogeneous boundaries, and/or nonhomogeneous differential equations. Since nonhomogeneous boundaries are easier, we shall treat these first.

Example 4–9. We wish to find the steady temperature of the problem given by Fig. 4–25.

The vertical axis is arbitrarily selected as the y-direction without altering the generality of the problem. Figure 4–25 may be divided into four problems, each one having one nonhomogeneous and three homogeneous boundary conditions. However, the simple transformation $\theta = T - T_\infty$ (or $\psi = T - T_0$) readily converts the nonhomogeneous condition h, T_∞ (or T_0) to a homogeneous condition. Using, for example, $\theta = T - T_\infty$, we may transform the problem of Fig. 4–25 to the problem of Fig. 4–26. This problem is expressible in terms of three suitable problems rather than four. Note that *the transformation of a nonhomogeneous boundary condition to a homogeneous boundary condition should not violate the physics of the boundary condition.* In other words, the only homogeneous form of $\pm k(\partial T/\partial n) = h(T - T_\infty)$ is $\pm k(\partial T/\partial n) = hT$, that of T_∞ is 0, and that of $\pm k(\partial T/\partial n) = q''$ is $(\partial T/\partial n) = 0$.

Hence the problem of Fig. 4–26 can be expressed in terms of the three problems shown in Fig. 4–27, such that the sum $\theta_1 + \theta_2 + \theta_3$ satisfies the differential equation and boundary conditions of Fig. 4–26. The solution of θ_1, θ_2, and

θ_3 may readily be obtained following the procedure of the previous examples, and will not be given here. The axes suitable to each problem are shown in

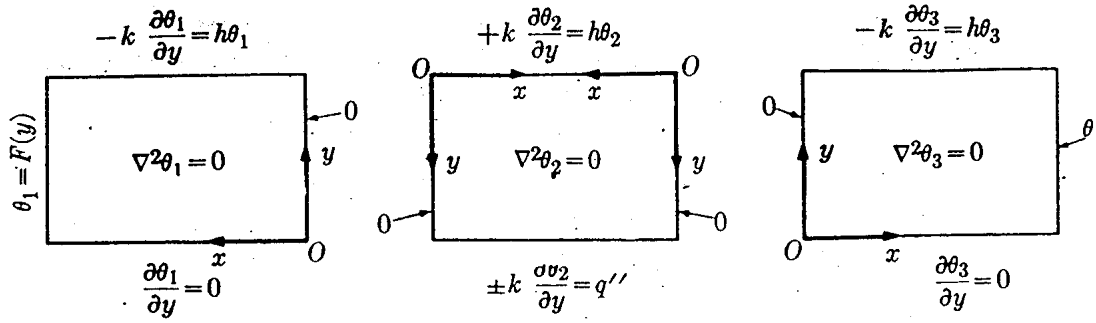

FIG. 4-27

Fig. 4-27. The *solutions of these problems can be added to each other only when they are expressed in a common coordinate system.*

The second type of nonhomogeneity, the nonhomogeneous differential equation, is related to internal energy generation.† The following example demonstrates the method of solution of these problems.

Example 4-10. Find the steady temperature of the electric heater of Example 2-1, given that the heat transfer coefficient is large.

The formulation of the problem in terms of Fig. 4-28 is

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+\frac{u'''}{k}=0, \qquad (4\text{-}124)$$

$$\frac{\partial\theta(0, y)}{\partial x}=0, \qquad \theta(L, y)=0, \qquad (4\text{-}125)$$

$$\frac{\partial\theta(x, 0)}{\partial y}=0, \qquad \theta(x, l)=0. \qquad (4\text{-}126)$$

The differential equation, being nonhomogeneous, is not separable.

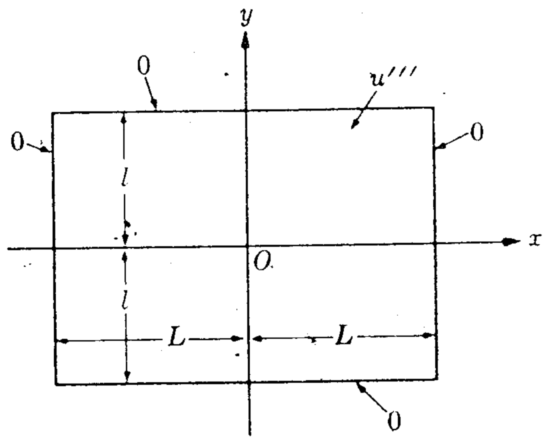

FIG. 4-28

The solution of the problem is now assumed to be

$$\theta(x, y)=\psi(x, y)+\phi(x) \qquad (4\text{-}127)$$

or

$$\theta(x, y)=\psi(x, y)+\phi(y). \qquad (4\text{-}128)$$

The use of either one of these forms is arbitrary in this case.

With the inclusion of the internal energy generation u''' in the formulation of the one-dimensional problem, $\phi(x)$ or $\phi(y)$, the differential equation to be satisfied by the two-dimensional problem, $\psi(x, y)$, can be made homogeneous. Thus $\psi(x, y)$ is suitable for separation of variables. However, the complete formulation of ϕ, say $\phi(x)$, and of $\psi(x, y)$ requires that the boundary conditions of these be specified. Here, $\phi(x)$ is assumed to satisfy the one-dimensional form of Eq. (4-125). Hence

$$\frac{d^2\phi}{dx^2} + \frac{u'''}{k} = 0; \qquad \frac{d\phi(0)}{dx} = 0, \quad \phi(L) = 0. \tag{4–129}$$

Then, combining Eqs. (4–124), (4–125), (4–126), (4–127), and (4–129), we find that $\psi(x, y)$ is satisfied by

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0, \tag{4–130}$$

$$\frac{\partial\psi(0, y)}{\partial x} = 0, \qquad \psi(L, y) = 0, \tag{4–131}$$

$$\frac{\partial\psi(x, 0)}{\partial y} = 0, \qquad \psi(x, l) = -\phi(x). \tag{4–132}$$

Thus the solution of the nonseparable problem $\theta(x, y)$ is reduced to that of the separable problem $\psi(x, y)$. The details of this solution are left to the reader. The result, including $\phi(x)$, is†

$$\frac{\theta(x, y)}{u'''L^2/k} = \frac{1}{2}\left[1 - \left(\frac{x}{L}\right)^2\right] - 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(\lambda_n L)^3}\left(\frac{\cosh \lambda_n y}{\cosh \lambda_n l}\right) \cos \lambda_n x, \tag{4–133}$$

where $\lambda_n L = (2n + 1)\pi/2$, $n = 0, 1, 2, \ldots$ Note that the foregoing procedure, being applicable to cylindrical and spherical geometries and to unsteady problems, is general.